

Angularly localized Skyrmions

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July 13, 2005

Abstract

Quantized Skyrmions with baryon numbers $B = 1, 2$ and 4 are considered and angularly localized wavefunctions for them are found. By combining a few low angular momentum states, one can construct a quantum state whose spatial density is close to that of the classical Skyrmion, and has the same symmetries. For the $B = 1$ case we find the best localized wavefunction among linear combinations of $j = \frac{1}{2}$ and $j = \frac{3}{2}$ angular momentum states. For $B = 2$, we find that the $j = 1$ ground state has toroidal symmetry and a somewhat reduced localization compared to the classical solution. For $B = 4$, where the classical Skyrmion has cubic symmetry, we construct cubically symmetric quantum states by combining the $j = 0$ ground state with the lowest rotationally excited $j = 4$ state. We use the rational map approximation to compare the classical and quantum baryon densities in the $B = 2$ and $B = 4$ cases.

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1 Introduction

The connection between the quantum and classical descriptions of a many-body system is an important but rather tricky one. In nuclei, the existence of a rotational band, a sequence of states whose energy increases with angular momentum j approximately as $\frac{\hbar^2}{2I}j(j+1)$, where I represents a moment of inertia, suggests the existence of a static intrinsic classical shape to the nucleus which is not spherically symmetric [1]. It is not obvious how this classical shape arises, and it is hard to predict the shape, but one can partially reconstruct it from the spectrum.

For a rigid body, the quantum states of various angular momenta are given by Wigner functions $D_{sm}^j(\alpha, \beta, \gamma)$, where α, β, γ are the Euler angles parametrizing the orientation, j is the total angular momentum and s, m its components with respect to the body-fixed and space-fixed third axis. Symmetries of the body constrain the possible s -values or combinations of s -values that can occur. A classically oriented state is a δ -function in the Euler angles. This can be obtained by taking an infinite linear combination of Wigner functions. For a body with symmetry, one would take a sum of δ -functions on a set of orientations related by symmetry (which are not distinguishable). Even if there is no fundamental rigid body to start with, one can consider these linear combinations. Thus, given a rotational band of states, one can construct a classically oriented state by taking an infinite linear combination of true quantum states of definite angular momentum. The properties of this oriented state (e.g. the particle density) would define the nature of the intrinsic state.

Something like this has been done in certain condensed matter situations. One may construct a classically oriented state when all that is rigorously available is quantum states labelled by angular momentum. Cooper et al. [2] have studied a model of rotating states of a Bose condensate trapped in a harmonic well (whose shape essentially makes the condensate two-dimensional). By numerically combining precise states over a range of angular momenta, they have shown that a condensate with vortices can be obtained. In the rotating frame, these vortices form a static array, and so are angularly localized despite the rotational invariance of the problem. The vortices do not really exist in any of the states of definite angular momentum, but they do in the combined state, and they can also be physically observed.

This localization depends on the system being large. Ideally, the moment of inertia should be almost infinite. In that case, the angular momentum states of different j are almost degenerate, and the angular localization may be achieved at almost no energy cost. (Similarly, an object with large mass can be spatially localized by taking a superposition of momentum states.)

Unfortunately, for nuclei, this is not always a realistic way to proceed. For large nuclei, like Hf^{170} , there are many states in a rotational band, and it is pretty clear that an intrinsic nuclear shape exists [1]. For smaller nuclei, however, at most a few low-lying states can be identified as forming a rotational band, and their energy separation is quite large because the moments of inertia are smaller. Not much is known about the wavefunctions of the states in the band, so it is hard to consider linear combinations. Instead it is better to postulate some intrinsic shape and fit its parameters to data. In this way, it is found, for example, that the Ne^{20} nucleus has a prolate deformation [3], but one cannot say it is exactly a prolate ellipsoid.

An alternative treatment of many-body systems can give angularly (and spatially) localized states much more easily. This is the approach based on an effective field theory, for example a Ginsburg–Landau description of a Bose condensate. Here, classical solutions of the field equation can naturally exhibit spatial order, for example an array of vortices. Because of the underlying symmetries, the classical solution is not unique, but is parametrized by collective coordinates describing, say, the center of mass position and angular orientation. Comparison with the previous discussion suggests that effective field theory can only be valid for large systems of many particles. To reconstruct quantum states of definite angular momentum, one may quantize the collective coordinates; this makes sense if the mass and the moment of inertia are of finite, but not infinite magnitude. A critical comparison of exact quantum states and classical solutions of an effective field theory has been carried out for quantum Hall ferromagnets by Abolfath et al. [4].

In this paper, we shall consider the Skyrme model and its connection with nuclei and their various angular momentum states. The Skyrme model is an effective field theory of pions, with a topological quantum number that can be identified with baryon number [5]. Skyrme’s original idea was that the model is justified because nuclei can be thought of as made up of a condensate of many light pions (with a topological winding). Recently, the justification is based on

the idea that each nucleon is made of N_c quarks, where $SU(N_c)$ is the gauge group of QCD, so a nucleus of baryon number B is made of a large number, $N_c B$, of quarks [6]. The Skyrme model has a semi-rigorous standing if N_c is large, but it is a controversial matter whether the physical value $N_c = 3$ is sufficiently large.

The classical Skyrme field equation, like that of the Ginsburg–Landau model, can be solved numerically and much is known about its minimal energy solutions (especially for pion mass equal to zero) [7]. Most importantly, the classical shapes of the solutions, and their symmetries, are known for values of B up to and beyond 20 (and work is underway to take account of the finite pion mass, which could have a qualitatively significant effect for $B \gtrsim 10$). These classical shapes could represent the intrinsic shapes of nuclei of modest size.

The shapes obtained have no obvious relation to shapes of nuclei as understood using other models, in particular, models based on point nucleons. For example, four-nucleon potential models are used to describe the α -particle, and the classical minimum occurs for a tetrahedral configuration of the nucleons [8]. In the Skyrme model, the solution of minimal energy with $B = 4$ has cubic symmetry.

Our aim in this paper is to bridge the gap between the classical Skyrmion shapes and the quantum states of nuclei. The traditional approach has been to quantize the collective coordinates of Skyrmions, seek the lowest energy states consistent with the allowed values of the angular momentum, and compare with the ground state properties of nuclei. This approach has some success in reproducing the known spins of nuclei, especially for even baryon number. More recently, a table of allowed angular momenta for the ground and first excited states of rotationally quantized Skyrmions has been constructed [9]. However, in these quantum states of definite angular momentum, the original Skyrmion shape information is sometimes completely lost.

We cannot consider an infinite linear combination of angular momentum states, as we expect that large angular momenta will lead to Skyrmion deformations, or if these are suppressed, then to infinite energy. Instead, here we shall consider an intermediate picture. By taking a small combination of low-lying angular momentum states, we partially reconstruct the shape of the classical Skyrmion solution. We shall optimize the angular localization of the Skyrmion within the limited combinations of states at our disposal. Such a

finite combination of states has finite energy (not necessarily very much higher than the ground state). If one could investigate theoretically (or experimentally, although this could be difficult) the same combination of angular momentum states in another nuclear model, one might see better the connection with the Skyrme model picture.

For the $B = 1$ case we apparently do not have the problem of orientation because a single Skyrmion has a spherically symmetric density. However, the Skyrmion still has rotational collective coordinates, and we will show that a particular combination of $j = \frac{1}{2}$ and $j = \frac{3}{2}$ states gives the most localized wave function. We also show that the ground state of the deuteron (the $j = 1$ state of the $B = 2$ Skyrmion), without an admixture of higher angular momentum states, retains the toroidal symmetry of the classical solution. Forest et al. have argued that not only the pure deuteron state, but also deuteron clusters within larger nuclei, show toroidal structure [10].

Finally, we shall show that a combination of $j = 0$ and $j = 4$ collective states of the $B = 4$ cubic Skyrmion gives a state close to the classically oriented Skyrmion. The same combination of $j = 0$ and $j = 4$ states in a four-nucleon potential model could be compared. Now it is inevitable that in the potential model, the state will have cubic symmetry because of the Wigner functions involved. In that sense our discussion is purely kinematic. However, some detailed properties of the state (density, currents) might show a close similarity with the Skyrme picture.

This paper is restricted to the $B = 1, 2, 4$ cases, and is organized as follows. Section 2 contains a review of the Skyrme model (for more details see [11]). In Section 3 we give an outline of the rational map approximation for Skyrmions, and its consequences, and recall the rational maps for the $B = 1$, $B = 2$ and $B = 4$ Skyrmions. In Section 4, we use the $j = \frac{1}{2}$ and $j = \frac{3}{2}$ quantum states of a $B = 1$ Skyrmion introduced in [12], and find the combination which gives the best localized wave function. In Sections 5 and 6 we use the rational maps introduced in Section 3 to find the “best” wavefunctions for $B = 2$ and $B = 4$ Skyrmions, respectively, and calculate the quantum baryon density in these states. In Section 7 we briefly discuss the implications of adding vibrational modes, and summarize our conclusions.

2 The Skyrme model

The Skyrme model [5] is an effective low energy theory of QCD attempting to treat pions, nucleons and nuclei. The topological soliton solutions arising from this model can be interpreted as baryons.

The model is defined by the Lagrangian

$$L = \int \left\{ \frac{F_\pi^2}{16} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{Tr}([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger][\partial^\mu U U^\dagger, \partial^\nu U U^\dagger]) \right\} d^3x, \quad (1)$$

where $U(t, \mathbf{x})$ is an $SU(2)$ -valued scalar field. F_π and e are parameters which can be scaled away by using energy and length units of $F_\pi/4e$ and $2/eF_\pi$, respectively. Thus, with the values of F_π and e as in [13] our units are related to conventional units via

$$\frac{F_\pi}{4e} = 5.58 \text{ MeV}, \quad \frac{2}{eF_\pi} = 0.755 \text{ fm}.$$

Introducing the $su(2)$ -valued right current $R_\mu = (\partial_\mu U)U^\dagger$ and using geometrical units, the Lagrangian (1) may be rewritten in the concise form

$$L = \int \left\{ -\frac{1}{2} \text{Tr}(R_\mu R^\mu) + \frac{1}{16} \text{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right\} d^3x. \quad (2)$$

The Euler–Lagrange equation which follows from (2) is the Skyrme equation

$$\partial_\mu \left(R^\mu + \frac{1}{4} [R_\nu, [R^\nu, R^\mu]] \right) = 0. \quad (3)$$

Static solutions are the stationary points (either minima or saddle points) of the energy function

$$E = \frac{1}{12\pi^2} \int \left\{ -\frac{1}{2} \text{Tr}(R_i R_i) - \frac{1}{16} \text{Tr}([R_i, R_j][R_i, R_j]) \right\} d^3x, \quad (4)$$

where we have introduced the additional factor $1/12\pi^2$ for convenience.

U , at fixed time, is a map from \mathbb{R}^3 into S^3 , the group manifold of $SU(2)$. However, the boundary condition $U \rightarrow 1$ implies a one-point compactification of space, so that topologically $U: S^3 \rightarrow S^3$, where the domain S^3 is identified with $\mathbb{R}^3 \cup \{\infty\}$. As the homotopy group $\pi_3(S^3)$ is \mathbb{Z} , maps between 3-spheres

are indexed by an integer, which is denoted by B . This integer is also the degree of the map U and has the explicit representation

$$B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \text{Tr}(R_i R_j R_k) d^3x. \quad (5)$$

As B is a topological invariant, it is conserved under continuous deformations of the field, including time evolution. This conserved topological charge Skyrme identified with baryon number.

Static fields of minimal energy, solving the Skyrme equation, are called multi-Skyrmions (Skyrmions, for short). They have been constructed numerically for B up to 22 [7], and the symmetries of these solutions have been identified. For $B = 1$ the Skyrme has spherical symmetry, and for $B = 2$ toroidal symmetry. It turns out that Skyrme have non-trivial discrete symmetries for $B > 2$. Solutions for negative B are obtained by the transformation $U \rightarrow U^\dagger$, which preserves the energy.

3 Rational map ansatz

In what follows, we will be using the rational map approximation to Skyrme [14]. Rational maps were first introduced into the theory of three-dimensional solitons by Jarvis [15], in the context of monopoles, but they prove to be very useful for Skyrme as well.

Rational maps are maps from $S^2 \rightarrow S^2$, whereas Skyrme are maps from $\mathbb{R}^3 \rightarrow S^3$. The idea in [14] is to identify the domain S^2 of the rational map with concentric spheres in \mathbb{R}^3 , and the target of the rational map S^2 with spheres of latitude on S^3 . A point in \mathbb{R}^3 can be parametrized by (r, z) ; r denotes radial distance and the complex variable z specifies the direction. Via stereographic projection z can be written in terms of usual polar coordinates θ and ϕ as $z = \tan(\theta/2)e^{i\phi}$. A rational map may be written as $R(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials in z . The degree of the rational map, N , is the greater of the algebraic degrees of the polynomials p and q . N is also the topological degree of the map (its homotopy class) as a map from $S^2 \rightarrow S^2$.

The point z on S^2 corresponds to the unit vector

$$\hat{\mathbf{n}}_z = \frac{1}{1 + |z|^2} (z + \bar{z}, i(\bar{z} - z), 1 - |z|^2). \quad (6)$$

Similarly, the value of the rational map R is associated with the unit vector

$$\hat{\mathbf{n}}_R = \frac{1}{1 + |R|^2} (R + \bar{R}, i(\bar{R} - R), 1 - |R|^2). \quad (7)$$

The ansatz for the Skyrme field, depending on a rational map $R(z)$ and a radial profile function $f(r)$, is

$$U(r, z) = \exp(ief(r)\hat{\mathbf{n}}_{R(z)} \cdot \boldsymbol{\tau}), \quad (8)$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ denotes the triplet of Pauli matrices, and $f(r)$ satisfies $f(0) = \pi$, $f(\infty) = 0$.

An $SU(2)$ Möbius transformation of z corresponds to a rotation in physical space; an $SU(2)$ Möbius transformation of R (i.e. on the target S^2) corresponds to an isospin rotation. Both these transformations of a rational map are symmetries of the Skyrme model, and both preserve N .

It can be verified that the baryon number for the ansatz (8) is given by

$$B = - \int \frac{f'}{2\pi^2} \left(\frac{\sin f}{r} \right)^2 \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2 \frac{2i dz d\bar{z}}{(1 + |z|^2)^2} r^2 dr. \quad (9)$$

$2i dz d\bar{z}/(1 + |z|^2)^2$ is equivalent to the usual 2-sphere area element $\sin \theta d\theta d\phi$. The angular part of the integrand,

$$\left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2 \frac{2i dz d\bar{z}}{(1 + |z|^2)^2}, \quad (10)$$

is precisely the pull-back of the area form $2i dR d\bar{R}/(1 + |R|^2)^2$ on the target sphere of the rational map R , so its integral is 4π times the degree N of the map. Therefore (9) simplifies to

$$B = \frac{-2N}{\pi} \int_0^\infty f' \sin^2 f dr = N. \quad (11)$$

An attractive feature of the rational map ansatz is that it leads to a simple energy expression which can be separately minimized with respect to the rational map R and the profile function f to obtain close approximations to the numerical, exact Skyrme solutions, and having the correct symmetries. (The numerical solutions are in fact best found by starting from the optimal rational map approximations.)

Indeed, using (8) we get the following expression for the energy (4):

$$E = \frac{1}{3\pi} \int_0^\infty \left(r^2 f'^2 + 2N \sin^2 f (f'^2 + 1) + \mathcal{I} \frac{\sin^4 f}{r^2} \right) dr. \quad (12)$$

Here \mathcal{I} denotes the purely angular integral

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i dz d\bar{z}}{(1+|z|^2)^2}, \quad (13)$$

which only depends on the rational map R . To minimize E , for maps of given degree N , one should first minimize \mathcal{I} over all maps of degree N . Then the profile function f , minimizing the energy (12), may be found by numerically solving a second order, ordinary differential equation with N and \mathcal{I} as parameters.

For $B = 1$, the rational map is $R(z) = z$, and this reproduces Skyrme's hedgehog ansatz [5], which is exactly satisfied by the $B = 1$ Skyrmion. For $B = 2$ and $B = 4$ the symmetries of the computed Skyrmions are $D_{\infty h}$ and O_h respectively, and in each case there is a unique rational map of the desired degree with the given symmetry, which also minimizes \mathcal{I} . They are, respectively,

$$R(z) = z^2, \quad R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \quad (14)$$

In all these cases, we have made a convenient choice of orientation in presenting the maps.

When quantizing the Skyrme field, we will be interested in the behavior of the wavefunction with respect to different orientations of the Skyrmion configurations. Consequently, all the information we need will be encoded in the angular dependence of the baryon density (10), which only depends on the rational map, and the profile function f will not be of much interest for our purposes.

4 $B = 1$ case

The $B = 1$ Skyrmion is spherically symmetric and takes the hedgehog form

$$U_0(\mathbf{x}) = \exp\{if(r)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}\}, \quad (15)$$

where $f(0) = \pi$ and $f(\infty) = 0$. If U_0 is the soliton solution, then $U = AU_0A^{-1}$, where A is an arbitrary constant $SU(2)$ matrix, is a static solution as well. But, in order to get solitons which are eigenstates of spin and isospin one needs to treat A as a collective coordinate. So substitute

$$U = A(t)U_0A^{-1}(t)$$

in the Lagrangian (1), where $A(t)$ is an arbitrary time-dependent $SU(2)$ matrix. The Lagrangian for A is [12]

$$L = -M + \lambda \text{Tr}(\partial_0 A \partial_0 A^{-1}), \quad (16)$$

where M is the soliton mass and λ is an inertia constant which may be found numerically.

The $SU(2)$ matrix A can be written as $A = a_0 + i\mathbf{a} \cdot \boldsymbol{\tau}$, with $a_0^2 + |\mathbf{a}|^2 = 1$. In terms of a_ξ ($\xi = 0, 1, 2, 3$) the Lagrangian (16) becomes

$$L = -M + 2\lambda \sum_{\xi=0}^3 (\dot{a}_\xi)^2, \quad (17)$$

and after the usual quantization procedure one gets the Hamiltonian

$$H = M + \frac{1}{8\lambda} \sum_{\xi=0}^3 \left(-\frac{\partial^2}{\partial a_\xi^2} \right). \quad (18)$$

Because of the constraint $a_0^2 + |\mathbf{a}|^2 = 1$, the operator $\sum_{\xi=0}^3 (\partial^2/\partial a_\xi^2)$ is to be interpreted as the laplacian ∇^2 on the 3-sphere. The wavefunctions can be expressed as traceless, symmetric, homogeneous polynomials in the a_ξ .

Using the isospin and spin operators

$$\begin{aligned} I_k &= \frac{1}{2}i \left(a_0 \frac{\partial}{\partial a_k} - a_k \frac{\partial}{\partial a_0} - \epsilon_{klm} a_l \frac{\partial}{\partial a_m} \right), \\ J_k &= \frac{1}{2}i \left(a_k \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_k} - \epsilon_{klm} a_l \frac{\partial}{\partial a_m} \right), \end{aligned} \quad (19)$$

Adkins, Nappi and Witten have found the normalized wavefunctions for neutron, proton and Δ -resonances [12]. The wavefunctions we require, having opposite I_3 and J_3 eigenvalues, are

$$\begin{aligned} |n, s_z = \frac{1}{2}\rangle &= \frac{i}{\pi} (a_0 + ia_3), \\ |p, s_z = -\frac{1}{2}\rangle &= -\frac{i}{\pi} (a_0 - ia_3), \\ |\Delta^-, s_z = \frac{3}{2}\rangle &= \frac{\sqrt{2}}{\pi} (a_0 + ia_3)^3, \\ |\Delta^0, s_z = \frac{1}{2}\rangle &= -\frac{\sqrt{2}}{\pi} (a_0 + ia_3)(1 - 3(a_1^2 + a_2^2)), \\ |\Delta^+, s_z = -\frac{1}{2}\rangle &= -\frac{\sqrt{2}}{\pi} (a_0 - ia_3)(1 - 3(a_1^2 + a_2^2)), \\ |\Delta^{++}, s_z = -\frac{3}{2}\rangle &= \frac{\sqrt{2}}{\pi} (a_0 - ia_3)^3. \end{aligned} \quad (20)$$

But none of these "pure" $j = 1/2$ and $j = 3/2$ states is the best localized wavefunction. This will instead be given by a superposition of the above states.

If we could take into account an infinite number of angular momentum states the most localized wavefunction would be the Dirac delta function, which may be expressed in the following form

$$\delta(\mu) = \sum_j (2j+1) \chi^j(\mu), \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (21)$$

where $\chi^j(\mu)$ is the character of the representation of dimension $(2j+1)$, and $a_0 = \cos \mu$. However, this wavefunction does not respect the Finkelstein–Rubinstein (FR) constraints [16], which in the case of one Skyrmion requires that the wavefunction is antisymmetric under $A \rightarrow -A$, thus ensuring that the quantized Skyrmion is a fermion. The sum in (21) must therefore be restricted to half-integer values of j , giving the total $\frac{1}{2}(\delta(\mu) - \delta(\mu - \pi))$.

For the $SU(2)$ group, the representation matrices are matrices of Wigner functions, i.e. for each j

$$\mathbf{D}^j = \begin{pmatrix} D_{jj}^j & \cdots & D_{j-j}^j \\ \vdots & \ddots & \vdots \\ D_{-jj}^j & \cdots & D_{-j-j}^j \end{pmatrix}.$$

In what follows we will be interested in the $j = 1/2$ and $j = 3/2$ cases, for which the Wigner functions take a concise form in terms of a_0, \dots, a_3 . The character χ^j is the trace of the above matrix, consequently let us write down the diagonal elements:

$$\begin{aligned} D_{1/2,1/2}^{1/2} &= a_0 + ia_3, \\ D_{-1/2,-1/2}^{1/2} &= a_0 - ia_3, \\ D_{3/2,3/2}^{3/2} &= (a_0 + ia_3)^3, \\ D_{1/2,1/2}^{3/2} &= (a_0 + ia_3)(1 - 3(a_1^2 + a_2^2)), \\ D_{-1/2,-1/2}^{3/2} &= (a_0 - ia_3)(1 - 3(a_1^2 + a_2^2)), \\ D_{-3/2,-3/2}^{3/2} &= (a_0 - ia_3)^3. \end{aligned} \quad (22)$$

If we truncate the sum (21) at $j = 3/2$ we get the following candidate for a well localized (normalized) wavefunction,

$$\Psi(a_0, a_1, a_2, a_3) = \frac{8}{\pi} \sqrt{\frac{2}{5}} \left(a_0^3 - \frac{3}{8} a_0 \right). \quad (23)$$

In terms of nucleon and Δ -resonance states this can be written as

$$\frac{1}{\sqrt{5}} \left(|\Delta^- \rangle - |\Delta^0 \rangle - |\Delta^+ \rangle + |\Delta^{++} \rangle - \frac{i}{\sqrt{2}} (|n \rangle - |p \rangle) \right), \quad (24)$$

with spins as in (20). A more general wavefunction of this type is

$$\Psi(a_0, a_1, a_2, a_3) = \frac{\sqrt{2}}{\pi} \left(\frac{5}{16} + \kappa + \kappa^2 \right)^{-1/2} (a_0^3 + \kappa a_0). \quad (25)$$

The maximum magnitude of Ψ at $a_0 = \pm 1$ occurs when $\kappa = -3/8$, confirming that this is the best localized wavefunction.

Another measure of how well the wavefunction is localized around $a_0 = \pm 1$ is given by the integral

$$\frac{2}{\pi^2} \left(\frac{5}{16} + \kappa + \kappa^2 \right)^{-1} \int_0^\pi a_0^2 |a_0^3 + \kappa a_0|^2 d\Omega. \quad (26)$$

Here $a_0 = \cos \mu$ and $d\Omega = 4\pi \sin^2 \mu d\mu$ is the measure of integration. After an easy calculation, we find that this integral is maximal when $\kappa = -1/4$, which is close to the result we got before. One more wavefunction worth considering is

$$\Psi = \frac{4}{\pi} \sqrt{\frac{2}{5}} a_0^3,$$

which is as well localized as the one with $\kappa = -3/8$ according to criterion (26), and rather simpler. It is the following combination of nucleon and Δ states:

$$\frac{1}{2\sqrt{5}} \left(|\Delta^- \rangle - |\Delta^0 \rangle - |\Delta^+ \rangle + |\Delta^{++} \rangle - 2\sqrt{2}i (|n \rangle - |p \rangle) \right). \quad (27)$$

These localized states are not physically important for isolated nucleons; however, they could be useful for modelling nucleons in interaction. Recent developments have shown that, for example, the deuteron is not formed from a proton and neutron only, but probably also contains some amount of Δ -resonances [17, 18]. Therefore, considering a superposition of states with different angular momenta is definitely physically meaningful. In [19] the deuteron was modelled by a bound state of Skyrmions in the attractive channel, where the relative orientation of the Skyrmions was chosen to maximize the attraction at short range. Such states could be approximated by the combined $j = 1/2$ and $j = 3/2$ states we have discussed here. The dependence of the force between two Skyrmions on their relative orientation is the classical analogue of the tensor force between nucleons, and it appears to automatically lead to an admixture of a Δ -resonance component to each nucleon.

5 $B = 2$ case

The $B = 2$ Skyrmion has $D_{\infty h}$ symmetry and a toroidal shape [20, 21, 22], and is used to describe the deuteron. We take the symmetry axis to be the third body-fixed axis, and the Skyrmion to be in its standard orientation if this coincides with the third Cartesian axis in space. In the rigid body approximation to quantization, the wavefunction is a function only of the rotational and isospin collective coordinates. (We ignore the translational collective coordinates, and set the momentum to zero.) To make an appropriate quantization we have to impose FR constraints, which tell us that the ground state has the quantum numbers $(i, j) = (0, 1)$, where i is the total isospin and j is the total spin. The wavefunction describing this deuteron state was obtained in [23]. Since $i = 0$, there is no dependence on the isospin collective coordinates, and the (normalized) state is

$$\Psi = \sqrt{\frac{3}{8\pi^2}} D_{0m}^1(\alpha, \beta, \gamma). \quad (28)$$

Here α , β and γ are the rotational Euler angles, $D_{0m}^1(\alpha, \beta, \gamma)$ is a Wigner function, and m is the third component of the space-fixed spin.

In [19], the analysis was extended to include one vibrational mode of the system, allowing the $B = 2$ toroidal Skyrmion to separate into two $B = 1$ Skyrmions. The wavefunction therefore includes a factor $u(\rho)$, the radial part of the deuteron wavefunction, which satisfies a radial Schrödinger equation on the interval $[\rho_0, \infty)$, where ρ_0 corresponds to the toroidal configuration. Here, however, we consider only the rigid body rotational states, and their angular dependence.

Since we are particularly interested in the spatial orientation, we treat states differing in m as different. The state we are looking for has to have the same symmetry properties as the classical solution. Consequently, the desired wavefunction is

$$\Psi = \sqrt{\frac{3}{8\pi^2}} D_{00}^1(\alpha, \beta, \gamma) = \sqrt{\frac{3}{8\pi^2}} \cos \beta, \quad (29)$$

which is axially symmetric both on the left and on the right (i.e. with respect to the body-fixed symmetry axis, and the x_3 -axis in space). Ψ has its maximum magnitude at $\beta = 0$ and $\beta = \pi$, corresponding to the Skyrmion in its standard orientation, and turned up-side down, which is classically indistinguishable after an isospin rotation.

Now, given the orientational quantum state (29) we may calculate the nucleon density, and find how quantum effects change the density of the classical configuration. We find the expression for the baryon density distribution $\rho_\Psi(\mathbf{x})$ in physical space (which is interpreted as nucleon density) by averaging the classical baryon density over orientations weighted with $|\Psi|^2$.

The density in the quantum state is therefore

$$\rho_\Psi(\mathbf{x}) = \int \mathcal{B}(D(A)^{-1}\mathbf{x}) |\Psi(A)|^2 \sin \beta \, d\alpha \, d\beta \, d\gamma. \quad (30)$$

Here A stands for the $SU(2)$ matrix parametrized by Euler angles α, β, γ , and $D(A)$ for the $SO(3)$ matrix associated to A via

$$D(A)_{ab} = \frac{1}{2} \text{Tr}(\tau_a A \tau_b A^\dagger). \quad (31)$$

As was already mentioned, the $B = 2$ rational map is $R(z) = z^2$, and this gives a good approximation to the $B = 2$ Skyrmon solution. It leads, using (9), to the following expression for the classical baryon density:

$$\mathcal{B}(r, z) = \frac{1}{\pi} \left(\frac{1 + |z|^2}{1 + |z|^4} |z| \right)^2 g(r), \quad (32)$$

where $g(r)$ is a radial function. $g(r)$ is unaffected by the quantum averaging, so we ignore it from now on. In terms of polar angles, the angular dependence of \mathcal{B} is given by

$$\mathcal{B} = \frac{1}{\pi} \frac{(1 + \tan^2(\frac{\theta}{2}))^2 \tan^2(\frac{\theta}{2})}{(1 + \tan^4(\frac{\theta}{2}))^2}, \quad (33)$$

where this is normalized to have angular integral equal to 2, the degree of the rational map.

To evaluate $\rho_\Psi(\mathbf{x})$ we first expand \mathcal{B} in terms of spherical harmonics $Y_{lm}(\theta, \phi)$:

$$\mathcal{B} = \sum_{l,m} c_{lm} Y_{lm}(\theta, \phi), \quad (34)$$

where, because of axial symmetry, there are only terms with $m = 0$. Although (34) is an infinite series it is a good approximation to take just the first two non-zero terms of the sum,

$$\mathcal{B} = c_{00} Y_{00}(\theta) + c_{20} Y_{20}(\theta), \quad (35)$$

as all the other terms contribute less than a 5% correction. Because the map R has degree 2, $c_{00} = 1/\sqrt{\pi}$; also, we find numerically that $c_{20} = -0.36$. Then $\mathcal{B}(D(A)^{-1}\mathbf{x})$ can be written as

$$\mathcal{B}(\tilde{\mathbf{x}}) = c_{00}Y_{00}(\tilde{\theta}) + c_{20}Y_{20}(\tilde{\theta}), \quad (36)$$

where $\tilde{\mathbf{x}} = D(A)^{-1}\mathbf{x}$ and similarly for $\tilde{\theta}, \tilde{\phi}$. Using the transformation properties of spherical harmonics under rotations,

$$Y_{lm}(\tilde{\theta}, \tilde{\phi}) = \sum_k D_{mk}^l(A)^* Y_{lk}(\theta, \phi), \quad (\text{no sum on } l), \quad (37)$$

the fact that $|\Psi|^2 = (3/8\pi^2)D_{00}^1(A)D_{00}^1(A)^*$, the orthogonality properties of the Wigner functions

$$\int D_{ab}^j(A)D_{cd}^{j'}(A)^* \sin\beta \, d\alpha \, d\beta \, d\gamma = \frac{8\pi^2}{2j+1} \delta^{jj'} \delta_{ac} \delta_{bd}, \quad (38)$$

and (in terms of the Wigner $3j$ symbols)

$$\int D_{ab}^j(A)D_{cd}^{j'}(A)D_{ef}^{j''}(A) \sin\beta \, d\alpha \, d\beta \, d\gamma = 8\pi^2 \begin{pmatrix} j & j' & j'' \\ a & c & e \end{pmatrix} \begin{pmatrix} j & j' & j'' \\ b & d & f \end{pmatrix}, \quad (39)$$

we find directly from (30) that the quantum probability distribution is

$$\rho_\Psi = c_{00}Y_{00} + \frac{2}{5}c_{20}Y_{20}. \quad (40)$$

This is an exact expression – no higher terms are present. We see that it resembles the classical distribution (35), but is more dominated by the first term. Thus, when quantum effects are included, the classical toroidal density remains, but is smoothed out to become more spherically symmetric.

6 $B = 4$ case

The minimal energy $B = 4$ solution has cubic symmetry; the region of high baryon density resembles a rounded cube with holes in the faces and at the centre [24]. We define the orthogonal body-fixed axes to be those passing through the face centres, and the standard orientation of the cube to be where these axes are aligned with the Cartesian axes in space. We shall again consider the Skyrmion as a rigid body, which means the configuration is not allowed to

vibrate. It was shown in [25] that in this case the ground state, representing the α -particle, has quantum numbers $i = 0$ and $j = 0$, with the (unnormalized) wavefunction $\Psi^{(0)} = 1$ being independent of the rotational and isospin collective coordinates. The first excited state has $i = 0$ and $j = 4$ and is [26]

$$\Psi_m^{(4)} = D_{4m}^4(\alpha, \beta, \gamma) + \sqrt{\frac{14}{5}} D_{0m}^4(\alpha, \beta, \gamma) + D_{-4m}^4(\alpha, \beta, \gamma). \quad (41)$$

In [26] the third component of the space-fixed spin, m , was arbitrary. The structure of (41) is required by the cubic symmetry with respect to body-fixed axes.

But just fixing m is not enough to make the wavefunction cubically symmetric both on the left and on the right, i.e. also with respect to space-fixed axes. To achieve this we need to take the following linear combination of the above wavefunctions:

$$\Psi^{(4)} = \Psi_4^{(4)} + \sqrt{\frac{14}{5}} \Psi_0^{(4)} + \Psi_{-4}^{(4)}. \quad (42)$$

The cubic symmetry in space is fairly obvious by analogy with (41), and can be verified as follows. First note that symmetry under 90° rotations about the x^3 -axis implies that all possible terms in (42) with m other than $\pm 4, 0$ vanish. To simplify the calculations a bit further we then introduce new variables

$$a = \cos\left(\frac{\beta}{2}\right) e^{\frac{1}{2}i\gamma} e^{\frac{1}{2}i\alpha}, \quad b = -\sin\left(\frac{\beta}{2}\right) e^{-\frac{1}{2}i\gamma} e^{\frac{1}{2}i\alpha}. \quad (43)$$

Obviously they satisfy $|a|^2 + |b|^2 = 1$. In terms of a and b , the $SU(2)$ orientation matrix parametrized by Euler angles α, β, γ is

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

In this notation the wavefunctions for different m take the following compact form:

$$\begin{aligned} \Psi_4^{(4)} &= a^8 + 14a^4b^4 + b^8 \\ \Psi_0^{(4)} &= \sqrt{70} \left(a^4\bar{b}^4 + \bar{a}^4b^4 + \frac{1}{40}(3 - 30(|a|^2 - |b|^2)^2 + 35(|a|^2 - |b|^2)^4) \right) \\ \Psi_{-4}^{(4)} &= \bar{a}^8 + 14\bar{a}^4\bar{b}^4 + \bar{b}^8. \end{aligned} \quad (44)$$

Therefore the wavefunction (42) in terms of a and b is

$$\begin{aligned}\Psi^{(4)} &= 2\text{Re}(a^8 + 14a^4b^4 + b^8) + 14(a^4\bar{b}^4 + \bar{a}^4b^4) \\ &\quad + \frac{7}{20}(3 - 30(|a^2| - |b^2|)^2 + 35(|a^2| - |b^2|)^4) .\end{aligned}\quad (45)$$

As expected, it is real. By acting on A with the generators of the cubic group:

$$\left(\begin{array}{cc} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{array} \right), \quad \left(\begin{array}{cc} \frac{1+i}{2} & \frac{1-i}{2} \\ -\frac{1+i}{2} & \frac{1-i}{2} \end{array} \right), \quad (46)$$

corresponding to a 90° rotation around a face of the cube, and a 120° rotation around a diagonal of the cube, we find the resulting transformations of (a, b) , and it is easy to check that $\Psi^{(4)}$ is cubically symmetric both on the left and on the right.

The wavefunction $\Psi^{(4)}$ has a positive maximum of $\frac{24}{5}$ at the identity, $(a, b) = (1, 0)$, and at all other elements of the (double cover of the) cubic group. This is as desired, as it corresponds to the Skyrmion having a high probability to be in its standard orientation. But $\Psi^{(4)}$ also has a negative minimum of $-\frac{104}{45}$, which gives a further local maximum of $|\Psi^{(4)}|^2$, at an orientation obtained by a 60° rotation around a diagonal of the cube, which is far from the standard orientation. We wish to suppress this.

We can do this by being a bit more sophisticated than in the $B = 2$ case. We still have the freedom of adding an arbitrary constant to the wavefunction. This means taking a superposition of the ground and first excited states, $\Psi^{(0)}$ and $\Psi^{(4)}$:

$$\Psi = \Psi^{(4)} + \kappa\Psi^{(0)} . \quad (47)$$

Here again we are interested in the nucleon density of the configuration. Our goal will be to adjust the constant κ to get a quantum distribution as close as possible to the classical one. As in the $B = 2$ case we define the quantum nuclear density via

$$\rho_\Psi(\mathbf{x}) = \int \mathcal{B}(D(A)^{-1}\mathbf{x}) |\Psi(A)|^2 \sin\beta \, d\alpha \, d\beta \, d\gamma , \quad (48)$$

where $\mathcal{B}(\mathbf{x})$ is the classical baryon density of the $B = 4$ Skyrmion in its standard orientation. Using the rational map (14), we find that \mathcal{B} has the angular dependence

$$\mathcal{B} = \frac{12}{\pi} |z|^2 (1 + |z|^2)^2 \frac{(z^4 \bar{z}^4 - z^4 - \bar{z}^4 + 1)}{(z^4 \bar{z}^4 + z^4 + 12z^2 \bar{z}^2 + \bar{z}^4 + 1)^2} . \quad (49)$$

Expressed in terms of polar angles,

$$\begin{aligned} \mathcal{B} = & \frac{12}{\pi} \tan^2 \left(\frac{\theta}{2} \right) \left(1 + \tan^2 \left(\frac{\theta}{2} \right) \right)^2 \\ & \times \frac{(\tan^8(\frac{\theta}{2}) - 2 \tan^4(\frac{\theta}{2}) \cos 4\phi + 1)}{(\tan^8(\frac{\theta}{2}) + 2 \tan^4(\frac{\theta}{2}) \cos 4\phi + 12 \tan^4(\frac{\theta}{2}) + 1)^2}, \end{aligned} \quad (50)$$

which may be expanded in the following form:

$$\mathcal{B} = d_0 Y_{00} + d_4 Z_4(\theta, \phi) + d_6 Z_6(\theta, \phi) + d_8 Z_8(\theta, \phi) + \dots \quad (51)$$

Here Z_4, Z_6 and Z_8 are the unique cubically symmetric combinations of spherical harmonics with, respectively $l = 4, 6$ and 8 :¹

$$\begin{aligned} Z_4 &= Y_{44} + \sqrt{\frac{14}{5}} Y_{40} + Y_{4-4}, \\ Z_6 &= Y_{64} - \sqrt{\frac{2}{7}} Y_{60} + Y_{6-4}, \\ Z_8 &= Y_{88} + \sqrt{\frac{28}{65}} Y_{84} + \sqrt{\frac{198}{65}} Y_{80} + \sqrt{\frac{28}{65}} Y_{8-4} + Y_{8-8}. \end{aligned} \quad (52)$$

The leading coefficient is $d_0 = 2/\sqrt{\pi}$ because the rational map has degree 4, and by numerical calculation we find that $d_4 = -0.28$, $d_6 = -0.032$ and $d_8 = 0.024$. Then, by a similar calculation as in the $B = 2$ case, normalizing the wavefunction and using the orthogonality properties of the Wigner functions, we find the following numerical result for the angular dependence of the quantum baryon density:

$$\rho_\Psi = d_0 Y_{00} + \frac{4}{2.56 + \kappa^2} \left\{ -(0.038 + 0.075\kappa) Z_4 - 0.006 Z_6 + 0.002 Z_8 \right\}, \quad (53)$$

which is again a finite sum, all the further terms being zero.

In (53), κ is not yet specified. Let us adjust it in such a way that the above distribution looks as close as possible to the classical one, i.e. let us maximize the coefficient of the $l = 4$ terms:

$$\frac{4}{2.56 + \kappa^2} (0.038 + 0.075\kappa).$$

¹These can be derived by combining the generating, cubically symmetric Cartesian polynomials $x^2 + y^2 + z^2$, $x^4 + y^4 + z^4$, $x^6 + y^6 + z^6$, and finding the combinations which satisfy Laplace's equation [27].

The maximum is at $\kappa \cong 1.17$, which leads to the following expression for the quantum baryon density:

$$\begin{aligned}\rho_\Psi &\cong 1.13Y_{00} - 0.13Z_4 - 0.006Z_6 + 0.002Z_8 \\ &\cong d_0Y_{00} + 0.46d_4Z_4 + 0.2d_6Z_6 + 0.1d_8Z_8.\end{aligned}\tag{54}$$

Thus in the $B = 4$ case, as in the $B = 2$ case, one can find a quantum state which localizes the Skymion close to its standard orientation, and which preserves the symmetry of the classical solution. However, the inclusion of quantum effects smoothes the classical baryon density, making it rather closer to spherically symmetric. Again, the effect is to approximately halve the leading non-constant harmonics, here with $l = 4$. If we considered the pure $j = 4$ state $\Psi^{(4)}$, we would get

$$\rho_{\Psi^{(4)}} \cong d_0Y_{00} + 0.2d_4Z_4 + 0.3d_6Z_6 + 0.15d_8Z_8,\tag{55}$$

which is much closer to spherically symmetric.

We can also find the energy of our state Ψ ; it is

$$E = \frac{1}{2.56 + \kappa^2} (2.56E_{j=4} + \kappa^2E_{j=0}) \cong 0.65E_{j=4} + 0.35E_{j=0},\tag{56}$$

so it is not as highly excited as a pure $j = 4$ state.

The combination of $j = 0$ and $j = 4$ states, Ψ , is a bit artificial as the quantum state of a free $B = 4$ Skymion, but would make sense if we were dealing with interacting Skymions (for example, when describing compound nuclei such as Be^8 , C^{12} in the Skyrme model equivalent of the α -particle model). Here we expect the relative orientations of the $B = 4$ subclusters to be rather precisely fixed when they are close together, so as to minimize their potential energy.

7 Conclusions

Three well-localized wavefunctions of the $B = 1$ Skymion have been considered and some of their advantages and physical implications have been discussed. The $B = 2$ and $B = 4$ minimal energy Skymion solutions have been quantized in such a way that the wavefunctions have the same symmetry properties as the classical Skymions (respectively, axial and cubic symmetry both on the left

and on the right), and angularly localized quantum states with shapes closest to the classical solutions have been found. In the $B = 4$ case, a superposition of two low-lying states of definite angular momentum needed to be considered. It is impossible to completely reproduce the shape of the classical solution this way. The quantum state necessarily smoothes out the classical baryon density, making it closer to being spherically symmetric.

All our results were obtained in the rigid body approximation, i.e. we did not allow the Skyrmions to vibrate. Considering the vibrational modes may be an interesting topic for future work. The vibrational modes for the $B = 2$ and $B = 4$ Skyrmions were calculated in [28, 29] and a qualitative analysis has been given for $B = 7$ [30]. The vibration frequencies obtained can be separated into those below and those above the breather mode, which is the oscillation corresponding to a change in scale size of the Skyrmion. In the modes below the breather, the Skyrmion tends to split up into individual $B = 1$ Skyrmions or small B Skyrmion clusters.

But treating the vibration modes as harmonic oscillators is not very accurate, since, as the minimal energy configuration separates into individual Skyrmions the potential flattens out. A more accurate treatment would involve estimating the inter-Skyrmion potential at intermediate and large separations. Thus it should not be expected that the inclusion of the zero point energy of harmonic vibrational modes will yield accurate results for masses, binding energies of states, etc.

The vibrational modes also couple to the rotational degrees of freedom, which complicates the analysis of the rotational and isospin wavefunctions [31].

Acknowledgements

O.M. thanks EPSRC for the award of a Dorothy Hodgkin Scholarship. N.S.M. thanks Nigel Cooper for a helpful discussion, and ECT*, Trento for hospitality.

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